

Quantum speed limit of a non-decoherent open system

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Abstract

We study the minimum time related to the quantum speed limit that characterizes the evolution of an open quantum system with the help of a simple model in the short and long time limits. We compare in particular the situation corresponding to a unique state and several states in the environment space. For short time intervals the results show a sensible difference in the behaviour of the system for different strengths of the interaction between the system and its environment.

Keywords: open quantum systems, quantum speed limit, time scales

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1 Introduction

The interval of time Δt over which a system moves by an amount of energy ΔE has a long history which starts with the time-energy uncertainty relation derived by Heisenberg [1] and was further generalized [2, 3, 4]. Since then a considerable amount of work has developed from thereon, see [5]. It led to the derivation of different approximate estimations of the time related to the so-called quantum speed limit t_{QSL} . The first important contributions concerning

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the optimisation of the time duration of quantum jumps have been realized by Mandelstam and Tamm [6] and followed by several other developments during the 90s [7, 8, 9, 10, 11]. The results concerned essentially closed systems.

During the following decade up to present days the subject developed further and extended to the case of open systems, see in particular [12, 13]. These developments were strongly motivated by the interest in the realization of fast electronic devices in which a given component interacts with its environment.

A large series of very recent work was devoted to the study of different properties which govern the behaviour of t_{QSL} in such systems such as the structure of the system space [17], the environment space [18], the importance of the interaction coupling with the environment and its optimization [19, 20, 21], the non-Markovian memory effects [24, 25], the initial state of the system [26], the role of a thermal environment [27], the influence of relativistic effects [28].

In the present work we aim to consider the behaviour of the time related to the lower bound of t_{QSL} and related quantities. We choose to perform a numerical analysis on a coherent open system described by a spin interacting with a bosonic environment. We first recall the expression of the time inequality in section 2 and develop the model and the analytic expression of the inequality in the framework of the model in section 3. Section 4 is devoted to the results. In section 5 we discuss and comment the results. Technical details are shown in the Appendices.

2 The time-energy relation

We introduce here the generalized energy-time relation to open quantum systems following the developments of Deffner and Lutz [12, 13].

Denote by $\hat{\rho}_S^{(0)}(t)$ the density operator of an isolated system S and by $\hat{\rho}_S(t) = Tr_E[\hat{\rho}(t)]$ the density operator of an open quantum system obtained as the trace over the states of an environment E coupled to S , where $\hat{\rho}(t)$ is the density operator of the closed system $S \oplus E$. The system S is characterized by its fidelity

$$F(\hat{\rho}_S^{(0)}(0), \hat{\rho}_S(t)) = Tr_S[(\hat{\rho}_S^{(0)}(0))^{1/2} \hat{\rho}_S(t) (\hat{\rho}_S^{(0)}(0))^{1/2}]^2 \quad (1)$$

which measures at time t the deviation of the open system from its initial state at the initial time at which the system does not interact with an environment.

The deviation can be expressed in terms of an angle [14, 15] which reads

$$B(\hat{\rho}_S^{(0)}(0), \hat{\rho}_S(t)) = \arccos(F(\hat{\rho}_S^{(0)}(0), \hat{\rho}_S(t))) \quad (2)$$

which is the generalization in the case of mixed states of the angle characterizing the overlap of two states in Hilbert space.

In the general case the time evolution of the system S is given by the master equation

$$d\hat{\rho}_S(t)/dt = D_t(\hat{\rho}_S(t)) \quad (3)$$

where D_t is a positive operator.

The average energy over a unit of time exchanged between S and E can be estimated by means of the quantity

$$\langle \Delta E(t) \rangle = \frac{1}{t} \int_0^t dt' e(t') \quad (4)$$

where $e(t)$ is taken as the smallest of the operator, trace or Hilbert-Schmidt norm of $D_t(\hat{\rho}_S(t))$:

$$\begin{aligned} \|D\|_{op}(t) &= \max[\lambda_i(t)] \\ \|D\|_{tr}(t) &= \sum_i [\lambda_i(t)] \\ \|D\|_{hs}(t) &= [\sum_i \lambda_i(t)^2]^{1/2} \end{aligned} \quad (5)$$

The $[\lambda_i(t)]$'s are the time-dependent eigenvalues of $D_t(\hat{\rho}_S(t))$ which in practice is the time derivative of the known density operator $\hat{\rho}_S(t)$, Eq.(3).

The minimum time corresponding to the quantum speed limit obeys the inequality given in ref. [13]:

$$t \geq t_{LB}(= t_{QSL}) \quad (6)$$

where the lower bound time is given by

$$t_{LB} = \hbar [\max(1/\|D\|_{op}(t), 1/\|D\|_{tr}(t), 1/\|D\|_{hs}(t))] |\cos(B(\rho_S^{(0)}(0), \rho_S(t))) - 1| \quad (7)$$

($\hbar = 1$ in what follows).

This time corresponds to a lower limit of the evolution time of the system. It is the smaller the larger the denominator corresponding to the energy exchange between S and E and the smaller the numerator, i.e. the larger the overlap between the initial state and the state of the system at time t .

3 Application of the inequality

3.1 Model

The system S is a spin which rotates around its Oz -axis and couples to a system of bosons.

The Hamiltonian \hat{H} reads

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{SE} \quad (8)$$

with

$$\begin{aligned} \hat{H}_S &= \omega \hat{J}_z \\ \hat{H}_E &= \beta b^+ b \\ \hat{H}_{SE} &= \eta (b^+ + b) \hat{J}^2 \end{aligned} \quad (9)$$

(b^+, b) are boson operators, ω is the rotation frequency of the system, β the quantum of energy of the bosonic oscillators and η the strength parameter of the coupling interaction between S and E .

In the present case H_{SE} commutes with H_S since \hat{J}_z and \hat{J}^2 commute in the common basis of states $[|jm\rangle]$ which are the eigenstates of both operators. It has been shown [16] that this class of systems does not lead to decoherence. In the subspace of \hat{J}_z limited to a unique spin state in j the projection of \hat{H}_{SE} on S space is diagonal in j_1 and reads

$$\langle j_1 | \hat{H}_{SE} | j_2 \rangle = \eta j_1 (j_1 + 1) (b^+ + b) \delta_{j_1, j_2} \quad (10)$$

As a consequence the density operator $\hat{\rho}_S(t)$ at time t is obtained by taking the trace over the environment states of the total Hamiltonian $\hat{\rho}(t)$ leading to

$$\hat{\rho}_S(t) = Tr_E \hat{\rho}(t) \quad (11)$$

whose matrix elements read

$$\rho_S^{j_1 m_1, j_2 m_2}(t) = \rho_0^{j_1 m_1, j_2 m_2}(t) \Omega_E(j_1, j_2, t) \delta(j_1, j_2) \quad (12)$$

with

$$\rho_0^{(0)j_1 m_1, j_2 m_2}(t) = \frac{e^{[-i\omega(m_1 - m_2)]t}}{(\hat{j}_1 \hat{j}_2)^{1/2}} \quad (13)$$

where $\hat{j}_i = 2j_i + 1$, m_1, m_2 are the spin projections on the quantization axis Oz . The bosonic environment contribution Ω_E can be put in the following form

$$\Omega_E(j_1, j_2, t) = 1/N(t) \sum_{n=0}^{n_{max}} \frac{1}{n!} \sum_{n', n''} \frac{E_{n, n'}(j_1, t) E_{n'', n}^*(j_2, t)}{[(n'!)(n''!)]^{1/2}} \quad (14)$$

where $N(t)$ is a normalization factor such that the trace of the density operator in the total space $S \oplus E$ is constant in time, $Tr\hat{\rho}(t) = 1$, $E_{n,n'}(t)$, $E_{n'',n}^*(t)$ are scalar quantities related to the coupling between S and E and n_{max} is the upper limit of the bosonic quantas present in the environment. An exact analytical expression of these quantities can be obtained by using an infinite Zassenhaus series [29] whose development is given in Appendix A. The explicit expressions of the polynomials $E_{n_1,n_2}(t)$ are developed in Appendix B.

By simple inspection of these expressions it can be seen that the non-diagonal element of $\rho_S^{j_1 m_1, j_2 m_2}(t)$ may cross zero with t , oscillate and never reach and stay at zero whatever the length of the time interval which goes to infinity. This is the signature of the fact that no decoherence of the system will be observable in this case.

3.2 The time-energy relation in the framework of the model

In the application of the model the spin of the system is supposed to stay in a unique j state, $j_1 = j_2 = j = 1/2$. The energy-time expression leads to the lower bound

$$t \geq \frac{\hbar |\cos(B(\rho_S^{(0)}(0), \rho_S(t)) - 1|}{\Delta_k(t)} \quad (15)$$

which can be worked out analytically. The index k corresponds to different choices of the integrant $\Delta(t) = \langle \Delta E(t) \rangle$ which is chosen as the the smallest one given by Eqs.(17 - 19). The fidelity takes the form

$$F^{j m_1, j m_2}(\hat{\rho}_S^{(0)}(0), \hat{\rho}_S(t)) = \frac{2\Omega_E(j, j; t)(1 + \cos(\omega t))}{\hat{j}^2} \quad (16)$$

which is a real quantity and consequently the expression of $B(\rho_S^{(0)}(0), \rho_S(t))$ is obtained by means of Eq.(2). Finally the energy denominator $max(\Delta_k(t))$ where $[k = (op), (tr), (hs)]$ is given by

$$\Delta_{op}(t) = 1/t \int_0^t max[|\lambda_i(t')|] dt' \quad (17)$$

where $\lambda_i(t)$, $i = 1, 2$ are the eigenvalues of the operator $d\hat{\rho}_S(t)/dt$ and similarly

$$\Delta_{tr}(t) = 1/t \int_0^t [|\lambda_1(t')| + |\lambda_2(t')|] dt' \quad (18)$$

and

$$\Delta_{hs}(t) = 1/t \int_0^t [|\lambda_1(t')|^2 + |\lambda_2(t')|^2]^{1/2} dt' \quad (19)$$

Details concerning the derivation of the fidelity are given in Appendix C.

4 Numerical applications

We use the present model in order to analyze the fidelity and the behaviour of the quantum speed limit under different physical conditions. We fix the energy parameters $\omega = 1$ and $\beta = 1$ and consider different strengths of the interaction $\eta_w = 0.1$ (weak coupling=W), $\eta_i = 1.0$ (intermediate coupling=I) and $\eta_s = 5.0$ (strong coupling=S). We choose different extensions of the bosonic environment, in practice $n_{max} = 0, 5, 10$. The system space is restricted to $j_1, j_2 = j = 1/2$. At time $t = 0$ the closed system is in a pure state, the state of lowest energy is

$$|\psi_S(0)\rangle = 1/\sqrt{2}(|1/2; 1/2\rangle + |1/2; -1/2\rangle) \quad (20)$$

The bosonic contribution at $t = 0$ is fixed by $\Omega_E(j, j; t = 0) = 1$.

We consider first the fidelity which is the essential ingredient in the r.h.s. of Eq.(15).

4.1 Evolution of the fidelity: Bures angle

a) Weak interaction: the fidelity shows an oscillating behaviour which indicates that the time dependent state vector $\psi_S(t)$ rotates with time. The evolution of $F(t)$ for different times t is the following:

For $\eta_w, n_{max} = 0$: $F(1) = 0.39$, $F(1.5) = 0.19$ and $F(t) \simeq 0$ for $t = 15, 20$. It oscillates for larger time intervals up to $t = 200$ with sizable amplitudes comparable to those of small time intervals.

The behaviour is qualitatively similar for $n_{max} = 5$ and $n_{max} = 10$. One observes a saturation effect from $n_{max} = 5$ to $n_{max} = 10$, the values of the fidelity are very similar to each other.

b) Intermediate interaction: the results are similar to those observed in the weak case. The amplitude oscillations start with a similar decrease but diminish more quickly than in the former case.

For $\eta_i, n_{max} = 0$: $F(1) = 0.29$, $F(1.5) = 10^{-4}$ and for $t = 10, 20$ $F(10) = 2.2 \times 10^{-2}$, $F(20) \simeq 0$.

For $n_{max} = 5$: $F(1) = 0.39$, $F(10) = 2.0 \times 10^{-2}$. $F(20) = 1.6 \times 10^{-1}$. The saturation effect from $n_{max} = 5$ to $n_{max} = 10$ is again seen here. The pseudo-period of oscillation is however somewhat shorter than in the weak case for $n_{max} = 5$ and $n_{max} = 10$. Oscillations persist over larger time intervals.

c) Strong interaction: the fidelity is large at the very beginning and gets quickly very small for early times. The oscillation period is much shorter than in the former cases. It goes on oscillating but with small amplitudes for larger time intervals.

For $\eta_s, n_{max} = 0$: $F(0.15) = 0.42$, $F(1) = 2.0 \times 10^{-5}$, $F(2) = 2.0 \times 10^{-6}$. It is similar for $n_{max} = 5$: $F(0.15) = 0.61$, $F(1) = 2.0 \times 10^{-3}$, $F(2) = 4.0 \times 10^{-4}$ and for $n_{max} = 10$: $F(0.15) = 0.61$, $F(1) = 2.0 \times 10^{-3}$, $F(2) = 4.1 \times 10^{-4}$. The fidelity goes on oscillating over larger period of times. The saturation effect is also present here.

In summary the fidelity is an oscillating function of the length of the time interval for any strength of the interaction. It is large for small times with an overall decrease in amplitude with increasing time. Its pseudo-period decreases with increasing interaction strength between the system and its environment. One observes a saturation effect with an increasing number of states in the environment.

4.2 The time dependent average energy of the system

The average energy over a unit time interval $\langle \Delta E(t) \rangle$ which enters the expression of the lower bound t_{LB} of t_{QSL} is defined in Eq.(4). The essential trend of its temporal evolution can be characterized as shown below.

a) Weak interaction: the average energy oscillates with a similar amplitude of the order of unity for small time intervals $t \in [0, 20]$ and $n_{max} = 0, 5, 10$. For large times ($t \in [50, 200]$) $\langle \Delta E(t) \rangle$ gets smaller and the oscillation amplitudes get reduced.

b) Intermediate interaction and strong interaction: the behaviour is qualitatively the same as in the weak case with smaller amplitudes and reduced oscillations over the time interval $t \in [0, 200]$.

4.3 The quantum speed bound

The lower bound t_{LB} defined in Eq.(7) is the physical quantity of interest in the present study. It depends on the behaviour of the fidelity (numerator) and the average energy per time unit present in the system (denominator). The analysis is done with the same set of parameters as those used for the fidelity.

a) Weak interaction: The lower bound t_{LB} is quite smaller than the evolution time interval, $t_{LB} \leq t$, whatever the extension of the bosonic space $n_{max} = 0, 5, 10$. It is very similar in magnitude in all cases. It is rather stationary for $t \leq 5$ and then increases slowly and quasi linearly, see Fig.1. For larger times ($t \in [100, 200]$) t_{LB} remains of the same order of magnitude and oscillates.

b) Intermediate interaction:

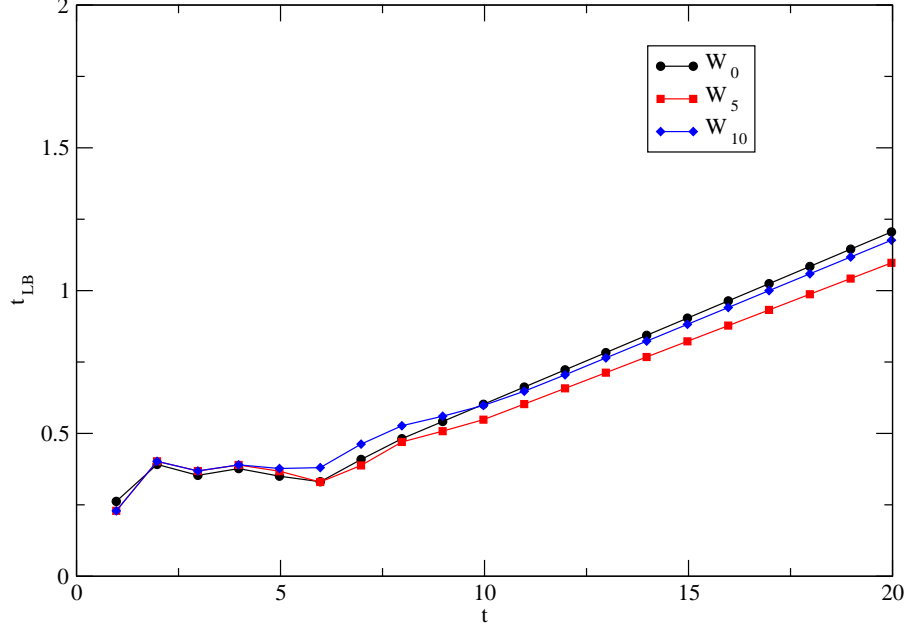


Figure 1: The lower bound t_{LB} : weak interaction, $n_{max} = 0, 5, 10$. See the text.

For $n_{max} = 0$ t_{LB} is quite smaller than the time interval t except for small $t \leq 5$ where it increases strongly with a steep slope, see Fig.2.

For $n_{max} = 5, 10$ one observes a saturation effect for which the values of t_{LB} come very close to each other. Otherwise the behaviour is the same as in the case $n_{max} = 0$, except for the fact that the magnitude of the oscillations are smaller, see Fig.2. This remains so for larger time intervals. The magnitudes of t_{LB} do not increase with time for $t \geq 5$ as it is the case in the weak regime, they tend even to decrease compared to their behaviour in the former regime.

c) Strong interaction:

Like in the intermediate strength case t_{LB} is somewhat larger for $n_{max} = 0$ than for $n_{max} = 5, 10$ but the oscillations in time follow rather closely in time and are more or less in phase with each other. The graph in Fig.3 shows sizable oscillations for $t \in [0, 20]$. For larger times ($t \in [20, 200]$) oscillations of t_{LB} remain but they are smaller than those observed for small times. Like in the case of the fidelity, the difference between the case $n_{max} = 5$ and $n_{max} = 10$ is negligible, the curves cannot be distinguished from each other.

Summing up:

- The lower bound of the time t_{LB} is generally closer to the time interval t for small time intervals than for long time intervals and shows oscillations

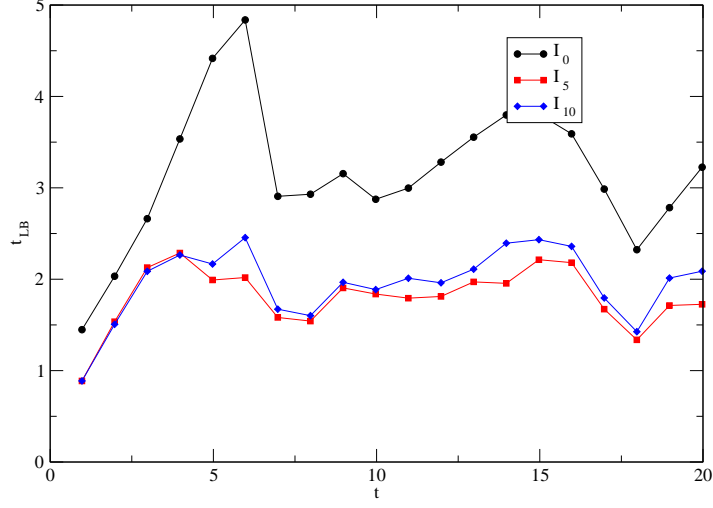


Figure 2: The lower bound t_{LB} : intermediate interaction, $n_{max} = 0, 5, 10$. See the text.

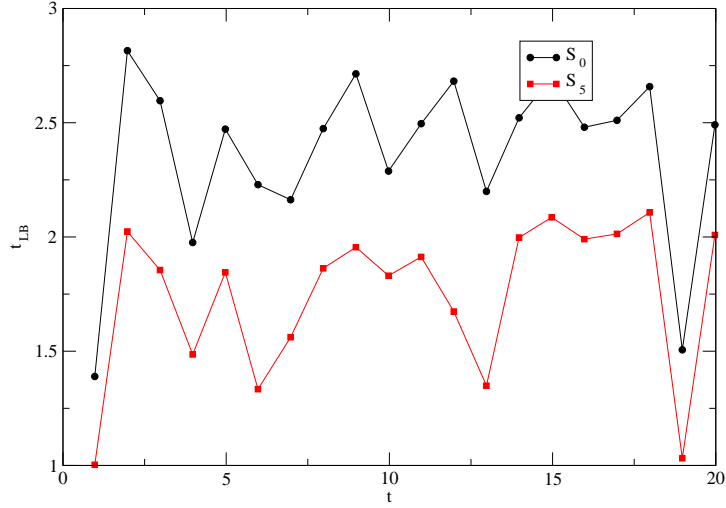


Figure 3: The lower bound t_{LB} : strong interaction, $n_{max} = 0, 5$. See the text.

which are essentially generated by the behaviour of the fidelity which appears in the numerator.

- For all three interaction strengths there is no qualitative difference between the behaviour of t_{LB} in the case where $n_{max} = 0$ and $n_{max} = 5, 10$ except over short time intervals. The fact that t_{LB} diminishes relative to increasing time intervals t in the case of an intermediate and strong interaction is due to the fact that in both cases the fidelity $F(t)$ decreases and the energy denominator $\langle \Delta E(t) \rangle$ varies slowly. This is not the case when the interaction is weak.
- The cases $n_{max} = 5$ and $n_{max} = 10$ are very similar, one observes a saturation effect with respect to an extension of the environment space in the size of t_{LB} .

5 Final comments

We used the time-energy inequality of Deffner and Lutz [12, 13, 21] in order to examine the behaviour of the speed limit in a non-decoherent open quantum system [22].

The time which characterizes the speed with which the open system evolves in time shows a lower bound t_{LB} which is governed by the fidelity $F(t)$ and an accumulated average energy per time unit $\langle \Delta E(t) \rangle$. Both quantities oscillate in time. This is due to the nature of the density matrix of the spin 1/2 system which shows oscillatory matrix elements.

We considered an environment space which contains a set n of non-interacting bosons with different numbers of excitation quanta and studied the evolution of the system for different values of the interaction strength between the system and the environment.

We first analyzed the behaviour of the fidelity $F(t)$ which leads to the Bures angle [14] and showed how the state vector of the open system evolves in time. It oscillates in the interval $[0, 1]$ for all values of the interaction with different amplitudes and periods depending on the strength chosen to be weak, intermediate and strong. The oscillatory character is induced by the structure of the density operator of the system. The oscillating average energy per time unit $\langle \Delta E(t) \rangle$ varies due to the energy exchange between the system and its environment.

The physically observable of interest is the lower time limit t_{LB} . Different strengths lead to different lower time bounds t_{LB} . In the case of a weak amplitude t_{LB} is close to the time interval t for small t .

For an intermediate and strong strengths of the interaction t_{LB} comes closer to the time interval t for small times. It gets smaller if the bosonic space contains several excited states. This is essentially due to the decrease of the numerator of the expression of t_{LB} governed by the fidelity, to some extent to the variation of the energy $\langle \Delta E(t) \rangle$. The size of t_{LB} is the smallest when the interaction is weak.

As it can be seen on the figures, one observes a rather systematic "saturation" effect in the sense that t_{LB} starts to be insensitive to an enlargement of the environment space, i.e. it remains the same when one increases the number of excitation quanta from 5 to 10 excitations. This is related to the fact that higher contributions to the quantity $\Omega_E(t)$ (Eq.14) which enters the expression of the fidelity (Eq.16) get small when the number of excitation quanta n increases.

It has been shown in a preceding work that the evolution of an open quantum system obeys the divisibility property which characterises a Markovian behaviour if the environment space reduces to a unique state [23]. It would be of interest to analyze the behaviour of t_{LB} in this specific case.

6 Appendix A: the Zassenhaus development

If $X = -i(t - t_0)(\hat{H}_S + \hat{H}_E)$ and $Y = -i(t - t_0)\hat{H}_{SE}$

$$e^{X+Y} = e^X \otimes e^Y \otimes e^{-c_2(X,Y)/2!} \otimes e^{-c_3(X,Y)/3!} \otimes e^{-c_4(X,Y)/4!} \dots \quad (21)$$

where

$$\begin{aligned} c_2(X, Y) &= [X, Y] \\ c_3(X, Y) &= 2[[X, Y], Y] + [[X, Y], X] \\ c_4(X, Y) &= c_3(X, Y) + 3[[[X, Y], Y], Y] + [[[X, Y], X], Y] + [[X, Y], [X, Y], \text{etc.} \end{aligned}$$

The series has an infinite number of term which can be generated iteratively in a straightforward way [30]. If $[X, Y] = 0$ the truncation at the third term leads to the factorisation of the X and the Y contribution. If $[X, Y] = c$ where c is a c -number the expression corresponds to the well-known Baker-Campbell-Hausdorff formula.

Here the expressions of the evolution operator $\exp(-iHt)$ can be rigorously worked out analytically and the series can be formally summed up to infinity for any time by means of analytic continuation arguments.

7 Appendix B: The bosonic content of the density operator

The expressions of the bosonic contributions to the density matrix $\rho_s^{j_1 m_1, j_2 m_2}(t)$ are given by

$$E_{n,n'}(j_1, t) = e^{-i\beta t} \sum_{n \geq n_2, n_3 \geq n_2} \sum_{n_3 \geq n_4, n' \geq n_4} (-i)^{n+n_3} (-1)^{n'+n_2-n_4} \frac{n!n'!(n_3!)^2 [\alpha_1(t)^{n+n_3-2n_2}] [\zeta_1(t)^{n_3+n'-2n_4}]}{(n-n_2)!(n_3-n_4)!(n_3-n_2)!(n'-n_4)!} e^{\Psi_1(t)} \quad (22)$$

and

$$E_{n'',n}^*(j_2, t) = e^{i\beta t} \sum_{n'' \geq n_2, n_3 \geq n_2} \sum_{n_3 \geq n_4, n \geq n_4} i^{n''+n_3} (-1)^{n+n_2-n_4} \frac{n''!n!(n_3!)^2 [\alpha_2(t)^{n''+n_3-2n_2}] [\zeta_2(t)^{n+n_3-2n_4}]}{(n''-n_2)!(n_3-n_2)!(n_3-n_4)!(n-n_4)!} e^{\Psi_2(t)} \quad (23)$$

The different quantities which enter $E_{n,n'}(t)$ are

$$\alpha_1(t) = \frac{\gamma(j_1) \sin \beta t}{\beta} \quad (24)$$

$$\zeta_1(t) = \frac{\beta[1 - \cos \gamma(j_1)t]}{\gamma(j_1)} \quad (25)$$

$$\gamma(j_1) = \eta j_1(j_1 + 1) \quad (26)$$

$$\Psi_1(t) = -\frac{1}{2} \left[\frac{\gamma^2(j_1) \sin^2(\beta t)}{\beta^2} + \frac{\beta^2(1 - \cos \gamma(j_1)t)^2}{\gamma^2(j_1)} \right] \quad (27)$$

and for $E_{n'',n}^*(t)$:

$$\alpha_2(t) = \frac{\gamma(j_2) \sin \beta t}{\beta} \quad (28)$$

$$\zeta_2(t) = \frac{\beta[1 - \cos \gamma(j_2)t]}{\gamma(j_2)} \quad (29)$$

$$\gamma(j_2) = \eta j_2(j_2 + 1) \quad (30)$$

$$\Psi_2(t) = -\frac{1}{2} \left[\frac{\gamma^2(j_2) \sin^2(\beta t)}{\beta^2} + \frac{\beta^2(1 - \cos \gamma(j_2)t)^2}{\gamma^2(j_2)} \right] \quad (31)$$

8 Appendix C: Calculation of the fidelity in the framework of the model

Define the operator

$$M(t) = [(\hat{\rho}_S^{(0)}(0))^{1/2}(\hat{\rho}_S(t))(\hat{\rho}_S^{(0)}(0))^{1/2}]^2 \quad (32)$$

Then the density operator in S space reads:

$$\hat{\rho}_S(t) = \frac{\hat{\rho}_S^{(0)}(t) * \Omega_E(j, j, t)}{(2j+1)} \quad (33)$$

For $j = 1/2$ the elements of the 2×2 matrix $M(t)$ can be worked out and read

$$M^{(m_1, m_2)}(t) = \text{Re}(\rho_S^{(1)}(t) + \rho_S^{(2)}(t)) \quad (34)$$

where

$$\begin{aligned} \rho_S^{(1)}(t) &= \rho_S^{(-1/2, -1/2)}(t) = 1 \\ \rho_S^{(2)}(t) &= \rho_S^{(-1/2, +1/2)}(t) = \exp(i\omega t) \end{aligned} \quad (35)$$

It is easy to construct the matrix $(M^{(m_1, m_2)}(t))^{1/2}$ whose matrix elements read

$$M^{(m_1, m_2)}(t)^{1/2} = \frac{[\Omega_E(j, j, t)(1 + \cos \omega t)]^{1/2}}{2^{1/2}(2j+1)} \quad (36)$$

Thereafter the fidelity $F(\hat{\rho}_S^{(0)}(0), \hat{\rho}_S(t))$ can be obtained and reads

$$F(\hat{\rho}_S^{(0)}(0), \hat{\rho}_S(t)) = \frac{2\Omega_E(j, j, t)(1 + \cos \omega t)}{(2j+1)^2} \quad (37)$$

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